Fourier series, sine series, cosine series

History: Fourier series were discovered by J. Fourier, a Frenchman who was a mathematician among other things. In fact, Fourier was Napolean's scientific advisor during France's invasion of Egypt in the late 1800's. When Napolean returned to France, he "elected" (i.e., appointed) Fourier to be a Prefect - basically an important administrative post where he oversaw some large construction projects, such as highway constructions. It was during this time when Fourier worked on the theory of heat on the side. His solution to the heat equation is basically what we teach in the last few weeks of sm212. The exception being that our understanding of Fourier series now is much better than what was known in the early 1800's and some of these facts, like Dirichlet's theorem, are covered as well.

Motivation: Fourier series, since series, and cosine series are all expansions for a function $f(x)$, much in the same way that a Taylor series is an expansion. Both Fourier and Taylor series can be used to approximate $f(x)$. There are at least three important differences between the two types of series. (1) For a function to have a Taylor series it must be differentiable[1], whereas for a Fourier series it does not even have to be continuous. (2) Another difference is that the Taylor series is typically not periodic (though it can be in some cases), whereas a Fourier series is *always* periodic. (3) Finally, the Taylor series (when it converges) always converges to the function $f(x)$, but the Fourier series may not (see Dirichlet's theorem below for a more precise description of what happens).

Definitions: Let $f(x)$ be a function defined on an interval of the real line. We allow $f(x)$ to be discontinuous but the points in this interval where $f(x)$ is discontinuous must be finite in number and must be jump discontinuities.

First, we discuss Fourier series. To have a Fourier series you must be given two things: (1) a "period" $P = 2L$, (2) a function $f(x)$ defined on an interval of length $2L$, usually we take $-L < x < L$ (but sometimes $0 < x < 2L$ is used instead). The **Fourier series of $f(x)$ with period $2L$** is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})],$$

[1]Remember the formula for the n-th Taylor series coefficient centered at $x = a$ - $a_n = \frac{f^{(n)}(a)}{n!}$?

where a_n and b_n are as in your blue USNA Math Tables[2].

Next, we discuss cosine series. To have a cosine series you must be given two things: (1) a "period" $P = 2L$, (2) a function $f(x)$ defined on the interval of length L, $0 < x < L$. The **cosine series of $f(x)$ with period $2L$** is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L}),$$

where a_n is given by

$$a_n = \frac{2}{L} \int_0^L \cos(\frac{n\pi x}{L}) f(x)\ dx.$$

(This formula is not in your USNA Math Tables.) The cosine series of $f(x)$ is exactly the same as the Fourier series of the **even extension** of $f(x)$, defined by

$$f_{even}(x) = \begin{cases} f(x), & 0 < x < L, \\ f(-x), & -L < x < 0. \end{cases}$$

Finally, we define sine series. To have a sine series you must be given two things: (1) a "period" $P = 2L$, (2) a function $f(x)$ defined on the interval of length L, $0 < x < L$. The **sine series of $f(x)$ with period $2L$** is

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L}),$$

where b_n is given by

$$b_n = \frac{2}{L} \int_0^L \sin(\frac{n\pi x}{L}) f(x)\ dx.$$

(This formula is also not in your USNA Math Tables.) The sine series of $f(x)$ is exactly the same as the Fourier series of the **odd extension** of $f(x)$, defined by

$$f_{odd}(x) = \begin{cases} f(x), & 0 < x < L, \\ -f(-x), & -L < x < 0. \end{cases}$$

[2] These formulas were not known to Fourier. To compute the Fourier coefficients a_n, b_n he used sometimes ingenious round-about methods.

One last definition: the symbol $\sim$ is used above instead of $=$ because of the fact that was pointed out above: the Fourier series may not converge to $f(x)$. Do you remember right-hand and left-hand limits from calculus 1? Recall they are denoted $f(x+) = \lim_{\epsilon\to 0,\epsilon>0} f(x+\epsilon)$ and $f(x-) = \lim_{\epsilon\to 0,\epsilon>0} f(x-\epsilon)$, resp.. The meaning of $\sim$ is that the series does necessarily not converge to the value of $f(x)$ at every point[3]. The convergence proprties are given by the theorem below.

Dirichlet's theorem[4]: Let $f(x)$ be a function as above and let $-L < x < L$. The Fourier series of $f(x)$,

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})],$$

where a_n and b_n are as in your blue USNA Math Tables, converges to

$$\frac{f(x+) + f(x-)}{2}.$$

In other words, the Fourier series of $f(x)$ converges to $f(x)$ only if $f(x)$ is continuous at x. If $f(x)$ is not continuous at x then then Fourier series of $f(x)$ converges to the "midpoint of the jump".

Examples: (1) If $f(x) = 2 + x$, $-2 < x < 2$, then the definition of L implies $L = 2$. Without even computing the Fourier series, we can evaluate it using Dirichlet's theorem.

Question: Using periodicity and Dirichlet's theorem, find the value that the Fourier series of $f(x)$ converges to at $x = 1, 2, 3$. (Ans: $f(x)$ is continuous at 1, so the FS at $x = 1$ converges to $f(1) = 3$ by Dirichlet's theorem. $f(x)$ is not defined at 2. It's FS is periodic with period 4, so at $x = 2$ the FS converges to $\frac{f(2+)+f(2-)}{2} = \frac{0+4}{2} = 2$. $f(x)$ is not defined at 3. It's FS is periodic with period 4, so at $x = 3$ the FS converges to $\frac{f(-1)+f(-1+)}{2} = \frac{1+1}{2} = 1$.)

The USNA Math Tables formulas enable us to compute the Fourier series coefficients a_0, a_n and b_n. (We skip the details.) These formulas give that the Fourier series of $f(x)$ is

$$f(x) \sim 4 + \sum_{n=1}^{\infty} -4\,\frac{n\pi\,\cos(n\pi)}{n^2\pi^2}\sin(\frac{n\pi x}{2}).$$

[3]This is what Fourier believed in the early 1800's when but we now know this is not always true.

[4]Pronounced "Dar-ish-lay".

The Fourier series approximations to $f(x)$ are

$$S_0 = 2,\ S_1 = 2 + \frac{4}{\pi}\sin(\frac{\pi x}{2}),\ S_2 = 2 + 4\,\frac{\sin(1/2\,\pi\,x)}{\pi} - 2\,\frac{\sin(\pi\,x)}{\pi},\ \ldots$$

The graphs of each of these functions get closer and closer to the graph of $f(x)$ on the interval $-2 < x < 2$. For instance, the graph of $f(x)$ and of S_8 are given below:

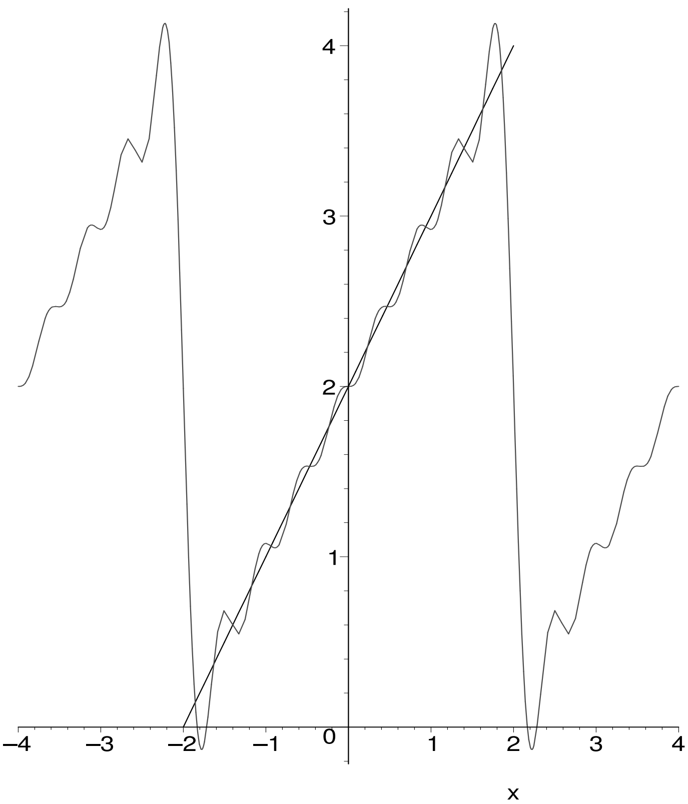

Figure 1: Graph of $f(x)$ and a Fourier series approximation of $f(x)$.

Notice that $f(x)$ is only defined from $-2 < x < 2$ yet the Fourier series is not only defined everywhere but is periodic with period $P = 2L = 4$. Also, notice that S_8 is not a bad approximation to $f(x)$.

(2) This time, let's consider an example of a cosine series. In this case, we take the piecewise constant function $f(x)$ defined on $0 < x < 3$ by

$$f(x) = \begin{cases} 1, & 0 < x < 2, \\ -1, & 2 \le x < 3. \end{cases}$$

We see therefore $L = 3$. The formula above for the cosine series coefficients gives that

$$f(x) \sim \frac{1}{3} + \sum_{n=1}^{\infty} 4 \frac{\sin(2/3\, n\pi)}{n\pi} \cos(\frac{n\pi x}{3}).$$

The first few partial sums are

$$S_2 = 1/3 + 2 \frac{\sqrt{3}\cos(1/3\, \pi\, x)}{\pi},$$

$$S_3 = 1/3 + 2 \frac{\sqrt{3}\cos(1/3\, \pi\, x)}{\pi} - \frac{\sqrt{3}\cos(2/3\, \pi\, x)}{\pi}, \ldots$$

As before, the more terms in the cosine series we take, the better the approximation is, for $0 < x < 3$. Comparing the picture below with the picture above, note that even with more terms, this approximation is not as good as the previous example. The precise reason for this is rather technical but basically boils down to the following: roughly speaking, the more differentiable the function is, the faster the Fourier series converges (and therefore the better the partial sums of the Fourier series will approximate $f(x)$). Also, notice that the cosine series approximation S_{10} is an even function but $f(x)$ is not (it's only defined from $0 < x < 3$).

For instance, the graph of $f(x)$ and of S_{10} are given below:

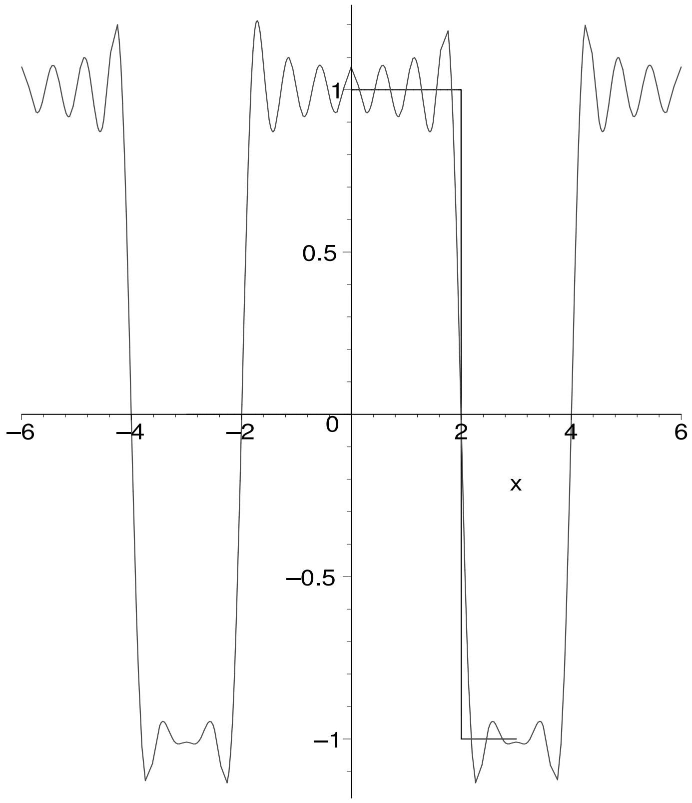

Figure 2: Graph of $f(x)$ and a cosine series approximation of $f(x)$.

(3) Finally, let's consider an example of a sine series. In this case, we take the piecewise constant function $f(x)$ defined on $0 < x < 3$ by the same expression we used in the cosine series example above.

Question: Using periodicity and Dirichlet's theorem, find the value that the sine series of $f(x)$ converges to at $x = 1, 2, 3$. (Ans: $f(x)$ is continuous at 1, so the FS at $x = 1$ converges to $f(1) = 1$. $f(x)$ is not continuous at 2, so at $x = 2$ the SS converges to $\frac{f(2+)+f(2-)}{2} = \frac{f(-2+)+f(2-)}{2} = \frac{-1+1}{2} = 0$. $f(x)$ is not defined at 3. It's SS is periodic with period 6, so at $x = 3$ the SS converges to $\frac{f(3-)+f_{odd}(-3+)}{2} = \frac{-1+1}{2} = 0$.)

The formula above for the sine series coefficients give that

$$f(x) = \sum_{n=1}^{\infty} 2\,\frac{\cos(n\pi) - 2\cos(2/3\,n\pi) + 1}{n\pi}\sin(\frac{n\pi x}{3}).$$

The partial sums are

$$S_2 = 2\,\frac{\sin(1/3\,\pi\,x)}{\pi} + 3\,\frac{\sin(2/3\,\pi\,x)}{\pi},$$

$$S_3 = 2\,\frac{\sin(1/3\,\pi\,x)}{\pi} + 3\,\frac{\sin(2/3\,\pi\,x)}{\pi} - 4/3\,\frac{\sin(\pi\,x)}{\pi}, \ldots$$

These partial sums S_n, as $n \to \infty$, converge to their limit about as fast as those in the previous example. Instead of taking only 10 terms, this time we take 40. Observe from the graph below that the value of the sine series at $x = 2$ does seem to be approaching 0, as Dirichlet's Theorem predicts. The graph of $f(x)$ with S_{40} is

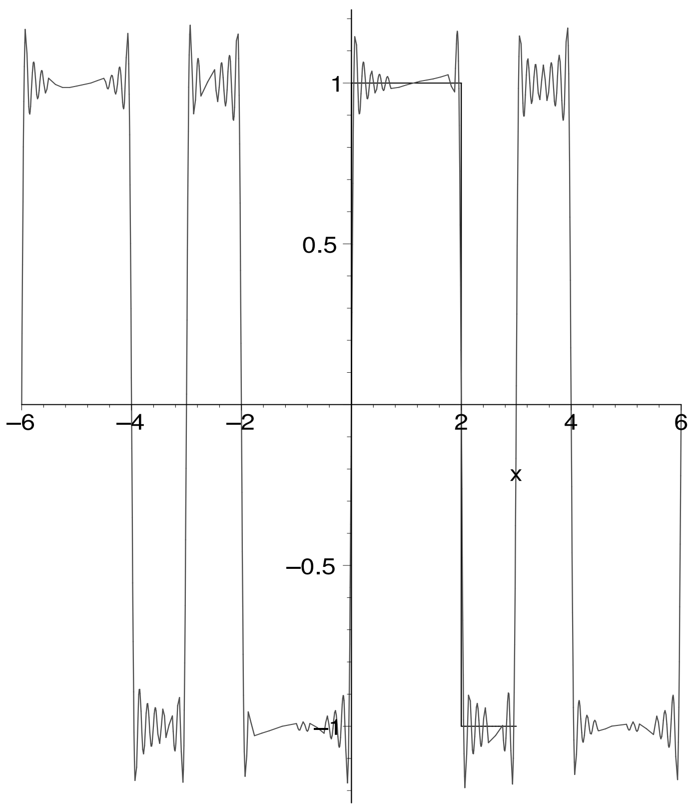

Figure 3: Graph of $f(x)$ and a sine series approximation of $f(x)$.